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RESEARCH REPORT No. CX-20

The Determination of the Scattering Potential from the Spectral Measure Function

Part III: Calculation of the Scattering Potential from the Scattering Operator for the One-Dimensional Schrödinger Equation

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THE DETERMINATION OF THE SCATTERING POTENTIAL FROM THE SPECTRAL MEASURE FUNCTION

Part III: Calculation of the Scattering Potential from the Scattering Operator for the One-Dimensional Schrödinger Equation

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Abstract

A procedure discussed in previous papers for the calculation of the scattering potentials from the spectral measure functions associated with certain eigenfunctions of the continuous spectrum of the perturbed Hamiltonian, has been adapted to the problem of obtaining the scattering potential from the scattering operator for the one-dimensional Schrödinger equation ($-\infty < x < \infty$). Examples are given to show how the procedure may be used.

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1. Introduction and Summary

Several writers [1], [2], [3] have recently discussed the problem of obtaining the scattering potential from the scattering phases for a variety of radial wave equations. Their work is characterized by the fact that they require the perturbed eigenfunctions to approach the unperturbed eigenfunctions at the origin. From this boundary condition and from specific properties of the unperturbed Hamiltonians and of the perturbation, they then obtain the Gelfand-Levitan equation [4], [5]. In the special cases discussed in [1], [2], [3] this equation was used to obtain the scattering potential in terms of the spectral measure function. By showing how the spectral measure function can be obtained from the scattering phases, it is then possible to derive the potential in terms of the phases.

In a previous paper [6] the authors provided a more abstract formulation of the problem of obtaining the scattering potential from the spectral measure function. It is the objective of the present paper to illustrate the use of the theorems presented there by considering the one-dimensional wave equation ($-\infty < x < \infty$). By showing how the spectral measure function can be obtained from the scattering operator S one can then calculate the potential from suitable elements of S. Actually this one-dimensional problem has already been treated by Kay [7] using properties of the one-dimensional wave equation discussed by Friedrichs [8]. The advantage of the more general approach of the present paper is that it seems to indicate a way to tackle the more interesting problem of three-dimensional scattering from a general (i.e., non-symmetric) potential.

We shall use the notation of [6] throughout, and indicate the identifications which must be made. For the representation in which K is to be triangular we shall take the usual x-representation, i.e., $q = x$. Furthermore, we shall work with eigenfunctions $< x|H,A;E,a >$ which approach a given eigenfunction $< x|H_o,A_o;E,a >$ of H_o when $x \rightarrow -\infty$, in contrast to previous authors who put boundary conditions on the eigenfunctions at the origin.

For conciseness we shall summarize the results below and prove them in subsequent sections.

First, let us define the reflection coefficient $b(k)$. Let us consider solutions $\Psi_k(x)$ of the equation

$$(1.1) \qquad \left[-\frac{d^2}{dx^2} + V(x)\right] \Psi_k(x) = k^2 \Psi_k(x)$$

which satisfy the condition that it represent the sum of an incident plane wave and an 'outgoing wave', i.e., which satisfy the integral equation

$$(1.2) \qquad \Psi_k^{out}(x) = e^{ikx} - \frac{i}{2k}\int_{-\infty}^{+\infty} e^{ik|x-x'|} V(x') \Psi_k^{out}(x')dx'.$$

The reflection coefficient $b(k)$ is defined by considering the following asymptotic form for $\Psi_k(x)$ when $k > 0$

$$(1.3) \qquad \lim_{x \to -\infty} \Psi_k^{out}(x) = e^{ikx} + b(k)\, e^{-ikx} \qquad (k > 0).$$

The reflection coefficient gives the amplitude of the plane wave which is reflected toward the left when an incident wave of unit amplitude moves toward the right.

Though we have defined $b(k)$ for $k > 0$ only, one can define this function for $k < 0$. One can show by analytic continuation

$$(1.4) \qquad b(-k) = b^*(k) \qquad \text{for } k \text{ real}$$

where the asterisk means complex conjugate.

We shall show that the potential $V(x)$ can be obtained from the knowledge of the reflection coefficient $b(k)$ alone if it satisfies certain sufficient conditions. (Some of these conditions are also necessary.)

Let $b(k)$ have the form

$$(1.5) \qquad b(k) = g(k)\, e^{-2ik\alpha} \qquad (\alpha > 0)$$

where

$$(1.6) \qquad g(k) = O(k^{-1}) \qquad \text{for } |k| \to \infty .$$

Further let the analytic continuation of $g(k)$ have positive imaginary residues r_j at a finite number of poles $k_j = i\,\tau_j$ $(j = 1,2, \ldots,n)$ where $\tau_j > 0$. Then if the equation

$$(1.7) \qquad \langle x|K|x'\rangle = -\langle x|\Omega|x'\rangle - \eta(x+x'+2\alpha)\int_{-(2\alpha+x')}^{x} \langle x|K|x''\rangle dx'' \langle x''|\Omega|x'\rangle$$

can be solved for $\langle x|K|x'\rangle$ $(x' < x)$ where

$$(1.8) \qquad \langle x|\Omega|x'\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k) e^{-ik(x+x')} dk - i\sum_j r_j e^{\tau_j(x+x'+2\alpha)} ,$$

and $\eta(x)$ is the usual Heaviside unit function

$$(1.8a) \qquad \eta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 , \end{cases}$$

then the scattering potential $V(x)$ is given by

$$(1.9) \qquad V(x) = 2\frac{d}{dx}\langle x|K|x\rangle .$$

Furthermore

$$(1.10) \qquad \langle x|K|x'\rangle \equiv V(x) \equiv 0 \qquad \text{when } x < -\alpha$$

and the point eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ will have the values $-\tau_j^2$. The reflection coefficient associated with $V(x)$ as obtained from (1.9) will be $b(k)$, as required by a consistent theory.

One will be able to obtain a unique set of eigenfunctions of $-\frac{d^2}{dx^2} + V(x)$ corresponding to the continuous spectrum

$$(1.11) \qquad \Psi_k(x) = e^{ikx} + \int_{-\infty}^{x} < x|K|x' > e^{ikx'}\, dx'$$

which satisfy the boundary condition

$$(1.12) \qquad \lim_{x \to -\infty} \Psi_k(x) = e^{ikx} .$$

The eigenfunctions corresponding to the point eigenvalues $-\tau_j^2$ denoted by $\Psi_{-i\tau_j}(x)$ are given by

$$(1.13) \qquad \Psi_{-i\tau_j} = e^{\tau_j x} + \int_{-\infty}^{x} < x|K|x' > e^{\tau_j x'}\, dx'.$$

The normalization of the eigenfunctions $\Psi_k(x)$ of the continuous spectrum given by (1.11) is

$$(1.14) \qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_k^*(x)\Psi_{k'}(x)dx + \frac{b(-k)}{2\pi}\int_{-\infty}^{+\infty} \Psi_{-k}^*(x)\, \Psi_{k'}(x)dx = \delta(k-k').$$

The bound states are orthogonal to eigenfunctions of the continuous spectrum. They satisfy the following orthogonality relation with respect to each other

$$(1.15) \qquad \int_{-\infty}^{+\infty} \Psi_{-i\tau_m}(x)\, \Psi_{-i\tau_j}(x)dx = i\delta_{mj}\, r_j^{-1}\, e^{-2\tau_j a} .$$

The eigenstates $\Psi_k(x)$ and $\Psi_{-i\tau_j}$ form a complete set and satisfy the completeness relationship

$$(1.16)\qquad \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty}\Psi_k(x)\Psi_k^*(x')dk+\int_{-\infty}^{+\infty}\Psi_k(x)b(-k)\Psi_{-k}^*(x')dk\right]$$

$$-\,i\sum_j\Psi_{-i\tau_j}(x)\Psi_{-i\tau_j}(x')r_je^{2\tau_ja}=\delta(x-x')\ .$$

2. The one-dimensional eigenfunctions; the scattering operator

The unperturbed Hamiltonian H_o is given by

$$(2.1)\qquad H_o^x=-\frac{d^2}{dx^2}\,,\qquad -\infty<x<+\infty,$$

where the superscript x on H_o^x signifies that the operator is expressed in the x-representation. It is convenient to use the eigenfunction $|H_o,A_o;E,a>$ given by

$$(2.2)\qquad <x|H_o,A_o;E,a>=\frac{(E)^{-1/4}}{2\sqrt{\pi}}\;e^{ia\sqrt{E}x}\,,$$

where a is restricted to having the values +1 or -1. The degeneracy variable a corresponds to the direction of the momentum. It is easily verified from (2.2) that

$$(2.3)\qquad <H_o,A_o;E',a'|H_o,A_o;E,a>=\int_{-\infty}^{+\infty}<H_o,A_o;E',a'|x>dx<x|H_o,A_o;E\ ,a\ >$$

$$=\delta(E-E')\ \delta_{a,a'}\ ,$$

where $\delta_{a,a'}$ is the usual Kronecker δ. It can also be verified that

$$(2.4) \qquad \sum_{a} \int_{0}^{\infty} |H_o, A_o; E, a > dE < H_o, A_o; E, a| = \eta(H_o),$$

where $\eta(H_o)$ is the identity operator in Hilbert space and maps remainder of the extended vector space into zero.

The perturbed Hamiltonian H is given by

$$(2.5) \qquad H = H_o + \varepsilon V.$$

Initially, we shall assume that $< x|V|x' >$ is generally not diagonal. Later we shall give conditions under which $< x|V|x' >$ is diagonal, i.e., has the form

$$(2.6) \qquad < x|V|x' > = V(x) \, \delta(x-x').$$

We shall assume, however, that $< x|V|x' >$ is such that a scattering operator exists.

The outgoing, and incoming eigenfunctions of H corresponding to the continuous spectrum satisfy the integral equations

$$(2.7) \qquad < x|H,A;E,a >_{\pm} = < x|H_o,A_o;E,a > \pm \frac{i}{2} \varepsilon(E)^{-1/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(\mp i \sqrt{E} \, |x-x'|) dx' \, < x'|V|x'' > dx'' < x''|H,A;E,a >_{\pm} .$$

In (2.7) we use the well-known result

$$(2.7a) \qquad < x|\gamma_{\pm}(E-H_o)|x' > = \pm \frac{i}{2} (E)^{-1/2} \exp(\mp i \sqrt{E} \, |x-x'|) .$$

The scattering operator and its inverse can be represented in the H_o-representation by (see [6], Part I, Eqs. (4.12), (4.20), and [6], Part II, Eqs. (3.7), (3.8))

$$< H_o,A_o;E,a|S|H_o,A_o;E',a' > = \delta(E-E') < a|S(E)|a' > \tag{2.8}$$

and

$$< H_o,A_o;E,a|S^{-1}|H_o,A_o;E',a' > = \delta(E-E') < a|S^{-1}(E)|a' >, \tag{2.9}$$

where

$$\begin{aligned} < a|S(E)|a' > &= \delta_{a,a'} - 2\pi i\varepsilon < H_o,A_o;E,a|V|H,A;E,a' >_- \\ &= \delta_{a,a'} - 2\pi i\varepsilon \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} < H_o,A_o;E,a|x > dx < x|V|x' > \\ &\qquad dx' < x'|H,a;E,a' >_- \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} < a|S^{-1}(E)|a' > &= \delta_{a,a'} + 2\pi i\varepsilon < H_o,A_o;E,a|V|H,A;E,a' >_+ \\ &= \delta_{a,a'} + 2\pi i\varepsilon \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} < H_o,A_o;E,a|x > dx < x|V|x' > \\ &\qquad dx' < x'|H,A;E,a' >_+ . \end{aligned} \tag{2.11}$$

For future reference we note the reciprocity theorem

$$\begin{aligned} < H_o,A_o;E,a|V|H,A;E,b >_- &= {}_+< H,A;E,a|V|H_o,A_o;E,b > = \\ &= < H_o,A_o;E,b|V|H,A;E,a >_+^* , \end{aligned} \tag{2.12}$$

where the asterisk means complex conjugate.

If $< x|V|x' >$ dies down sufficiently rapidly, the asymptotic forms of $< x|H,A;E,a >_{\pm}$ are related to $< a|S(E)|b >$ and $< a|S^{-1}(E)|b >$ by

$$\lim_{x \to \pm\infty} < x|H,A;E,a >_{-} = < x|H_o,A_o;E,a > + < x|H_o,A_o;E, \pm 1 > \left[< \pm 1|S(E)|a > - \delta_{\pm 1,a} \right] \tag{2.13}$$

$$\lim_{x \to \pm\infty} < x|H,A;E,a >_{+} = < x|H_o,A_o;E,a > + < a|H_o,A_o;E, \mp 1 > \left[< \mp 1|S^{-1}(E)|a > -\delta_{\mp 1,a} \right], \tag{2.14}$$

as can be shown from (2.10) and (2.11).

3. Expression for the weight operator in terms of the scattering operator

We shall now express in terms of S and S^{-1} the operators W_c, $M_{\pm}$ associated with those operators U such that the eigenfunctions of H given by

$$< x|H,A;E,a > = < x|U|H_o,A_o;E,a > \tag{3.1}$$

satisfy the boundary condition

$$\lim_{x \to -\infty} < x|H,A;E,a > - < x|H_o,A_o;E,a > = 0 \tag{3.2}$$

or, symbolically,

$$\lim_{x \to -\infty} < x|H,A;E,a > = < x|H_o,A_o;E,a > . \tag{3.2a}$$

Since we shall work in the continuous spectrum of H, we shall drop the factor

$\eta(H_o)$ for the time being, since it will always act like the identity operator.

From $U = U_- M_-$ we can write

(3.3) $$< x|H,A;E,a > = \sum_{a'} < x|H,A;E,a' >_- \quad < a'|\mu_-(E)|a > ,$$

where $< a|\mu_-(E)|a' >$ is given by (cf. [6], Part I, Eq. (5.5))

(3.4) $$< H_o,A_o;E,a|M_-|H_o,A_o;E',a' > = \delta(E-E') < a|\mu_-(E)|a' > .$$

From Eqs. (3.2a) and (2.13) we obtain an equation for $< a|\mu_-(E)|a' >$, namely

(3.5) $$< x|H_o,A_o;E,a > = \sum_{a'} < x|H_o,A_o;E,a' > < a'|\mu_-(E)|a >$$
$$+ < x|H_o,A_o;E,-1 > \sum_{a'} \Big[< -1|S(E)|a' > - \delta_{-1,a'} \Big] < a'|\mu_-(E)|a >.$$

On multiplying (3.5) through by $< H_o,A_o;E'',a''|x >$ and integrating with respect to x, and then using the orthogonality relation (2.3), one finds that

(3.6) $$\delta(E''-E)\delta_{a'',a} = \delta(E''-E) \sum_{a'} \delta_{a'',a'} < a'|\mu_-(E)|a >$$
$$+ \delta(E''-E)\delta_{a'',-1} \sum_{a'} \Big[< -1|S(E)|a' > -\delta_{-1,a'} \Big] < a'|\mu_-(E)|a >,$$

which on renaming some of the variables, may be rewritten as

(3.7) $$\delta_{a,a'} = < a|\mu_-(E)|a' > + \delta_{a,-1} \sum_{a''} \Big[< -1|S(E)|a''> -\delta_{-1,a''} \Big] < a''|\mu_-(E)|a' >.$$

Eq. (3.7) may be regarded as an equation for the matrix $< a|\mu_-(E)|a' >$. The solution of (3.7) can be shown to be

$$(3.8) \quad < a|\mu_-(E)|a' > = \begin{pmatrix} 1 & 0 \\ \frac{-< -1|S(E)|+1 >}{< -1|S(E)|-1 >} & \frac{1}{< -1|S(E)-1 >} \end{pmatrix},$$

where we adopt the convention that the first row and column are labeled by +1 and the second by -1, i.e., $< +1|\mu_-(E)|-1 > = 0$. Eq. (3.8) gives us M_- in the H_0-representation.

From [6], Part I, Eq. (5.15) we have

$$(3.9) \qquad W_c = M_-^{-1} \; M_-^{*-1} \; .$$

To find W_c, therefore, we shall have to obtain M_-^{-1} and M_-^{*-1}. We shall express these operators in terms of the H_0-representation. Let us define $< a|\mu_-^{-1}(E)|a' >$ by

$$(3.10) \qquad < H_0,A_0;E,a|M_-^{-1}|H_0,A_0;E',a' > = \delta(E-E') < a|\mu_-^{-1}(E)|a' >.$$

From $M_-^{-1}M_- = M_-M_-^{-1} = \eta(H_0)$, (where we recall $\eta(H_0)$ is the identity in Hilbert space) it can be shown that for fixed E the quantities $< a|\mu_-^{-1}(E)|a' >$ are just the elements of the matrix which is the inverse of the matrix whose elements are $< a|\mu_-(E)|a' >$. It is then shown that

$$(3.11) \quad < a|\mu_-^{-1}(E)|a' > = \begin{pmatrix} 1 & 0 \\ < -1|S(E)|+1 > & < -1|S(E)|-1 > \end{pmatrix}.$$

Let us now define $< a|M_-^{*-1}(E)|a' >$ by

(3.12) $$< H_o,A_o;E,a|M_-^{*-1}|H_o,A_o;E',a' > = \delta(E-E') < a|\mu_-^{*-1}(E)|a' > .$$

From the fact that $M_-^{*-1} = (M^*)^{-1} = (M^{-1})^*$ is the Hermitian adjoint of M^{-1} we have

(3.13) $$< a|\mu_-^{*-1}(E)|a' > = < a'|\mu_-^{-1}(E)|a >^* .$$

If, as in [6] we define $< a|\omega_c(E)a' >$ by

(3.14) $$< H_o,A_o;E,a|W_c|H_o, A_o;E',a' > = \delta(E-E') < a|\omega_c(E)|a' > ,$$

we have from (3.9)

(3.15) $$< a|\omega_c(E)|a' > = \sum_{a''} < a|\mu_-^{-1}(E)|a''> < a''|\mu_-^{*-1}(E)|a' >,$$

or, on using (3.13) and (3.11)

(3.16) $$< a|\omega_c(E)|a' > = \begin{pmatrix} 1 & < -1|S(E)|+1 >^* \\ < -1|S(E)|+1 > & 1 \end{pmatrix} .$$

In deriving (3.16) from (3.15) we have used the relation

(3.17) $$| < -1|S(E)|-1 > |^2 + | < -1|S(E)|+1 > |^2 = 1,$$

which follows from the fact that $S^* = S^{-1}$, i.e., $SS^* = \eta(H_o)$.

For the sake of completeness, we shall also give the expression for $< a|\mu_+(E)|a' >$ defined by

(3.18) $$< x|H,A;E,a > = \sum_{a'} < x|H,A;E,a' >_+ < a'|\mu_+(E)|a > ,$$

where $< x|H,A;E,a >$ satisfies the boundary condition (3.2a). The expression can be shown to be

$$(3.19) \qquad < a|\mu_+(E)|a' > \; = \begin{pmatrix} \dfrac{1}{< +1|S^{-1}(E)|+1 >} & \dfrac{- < +1|S^{-1}(E)|-1 >}{< +1|S^{-1}(E)|+1 >} \\ 0 & 1 \end{pmatrix} .$$

4. The complex energy plane

In this section we consider the analytic continuation of various quantities introduced above, and for this purpose we introduce the complex E-plane which shall have a cut along the positive real axis. The results obtained here will be useful in later sections.

Let us first introduce the function $< x|H_o,A_o;E,a >^+$. This function is defined as the analytic continuation of $< x|H_o,A_o;E,a >$ when E is on the upper part of the cut. From (2.2), we see that when E is in the lower part of the cut

$$(4.1) \qquad < x|H_o,A_o;E,a >^+ \; = \; -i < x|H_o,A_o;E,-a > \quad .$$

One can also introduce $< x|H_o,A_o;E,a >^-$ which defined as being equal to $< x|H_o,A_o;E,a >$ when E is on the lower side of the cut. Hence, from (4.1)

$$(4.2) \qquad < x|H_o,A_o;E,a >^- \; = \; i < x|H_o,A_o;E,-a >^+$$

and

$$(4.3) \qquad < x|H_o,A_o;E,a >^- \; = \; i < x|H_o,A_o;E,-a >$$

when E is on the upper side of the cut.

We can also introduce $< H_o,A_o;E,a|x >^+$ which equals $< H_o,A_o;E,a|x >$ when E is on the upper side of the cut. Then since

$$< H_o,A_o;E,a|x > \; = \; < x|H_o,A_o;E,-a > \tag{4.4}$$

we have

$$< H_o,A_o;E,a|x >^+ \; = \; < x|H_o,A_o;E,-a >^+ \; . \tag{4.5}$$

Similarly one can define $< H_o,A_o;E,a|x >^-$ as being the analytic continuation of $< H_o,A_o;E,a|x >$ when E is at the bottom of the cut, where from (4.4) it can be shown that

$$< H_o,A_o;E,a|x >^- \; = \; < x|H_o,A_o;E,-a >^- \; . \tag{4.6}$$

Now we define $< x|H,A;E,a >_{\pm}^{+}$ as the analytic continuation of $< x|H,A;E,a >_{\pm}$ when E is on the top of the cut. From the integral equation (2.7) it is clear that when E is at the bottom of the cut we have

$$< x|H,A;E,a >_{\pm}^{+} = -i < x|H_o,A_o;E,-a > \mp \frac{i}{2}\, \varepsilon(E)^{-1/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dx''\, e^{\pm i\sqrt{E}\,|x-x'|} \; < x'|V|x''> < x''|H,A;E,a >_{\pm}^{+} \tag{4.7}$$

from which it can be shown that

$$< x|H,A;E,a >_{\pm}^{+} \; = \; -i < x|H,A;E,-a >_{\mp} \tag{4.8}$$

when E is on the lower part of the cut.

Likewise one can introduce $< x|H,A;E,a >^{-}_{\pm}$ which is the analytic continuation of $< x|H,A;E,a >_{\pm}$ defined on the lower part of the cut. It is clear from (4.8) that

$$(4.9) \qquad < x|H,A;E,a >^{-}_{\pm} = i < x|H,A;E,-a >^{+}_{\mp}$$

and that when E is at the top of the cut

$$(4.10) \qquad < x|H,A;E,a >^{-}_{\pm} = i < x|H,A;E,-a >_{\mp} \;.$$

The analytic continuation of $< a|S(E)|a' >$ when E is on the top of the cut will be denoted by $< a|S(E)|a' >^{+}$. From (2.10), (4.8), (4.5), (4.1), (2.11), (2.12) we see that when E is at the bottom of the cut

$$(4.11) \qquad < a|S(E)|a' >^{+} = \delta_{a,a'} + 2\pi i \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx'$$

$$\cdot < H_o,A_o;E,-a|x > < x|V|x' > < x'|H,A;E,-a' >_{+}$$

$$= < -a|S^{-1}(E)|-a' > = < -a'|S(E)|-a >^{*} ,$$

where, as usual, the asterisk means complex conjugate. We denote by $< a|S(E)|a' >^{-}$ the analytic continuation of $< a|S(E)|a' >$ when E is at the bottom of the cut. It can be shown that when E is at the top of the cut

$$(4.12) \qquad < a|S(E)|a' >^{-} = < -a|S^{-1}(E)|-a' > = < -a'|S(E)|-a >^{*} .$$

5. Extension of the unperturbed Hamiltonian. Diagonal form for the scattering potential.

In [6] we saw that it is necessary to extend the definition of the operator H_o to a space larger than Hilbert space in order that the extended operator H_o have the same spectrum as H. In [6], Part II the extension was carried out by introducing 'eigenfunctions' of the extended operator H_o. Here we specialize this procedure to the present case.

In Hilbert space H_o is defined by the way it acts in the x-representation, namely,

$$(5.1) \qquad H_o^x = - \frac{d^2}{dx^2} .$$

We shall require that the extended space, H_o, be also given by (5.1). In the notation of [6], Part II, H_o is then an x-extended operator. With this extension the potential V has the desirable property that it is diagonal in the x-representation, as we shall show shortly. First, however, we shall obtain the additional eigenfunctions necessary to complete the extended space. In accordance with [6], Part II we shall look for solutions of

$$(5.2) \qquad H_o|H_o,A_o;E,a> = E|H_o,A_o;E,a>$$

for values of E in the vicinity of each E_i which is a point eigenvalue of H such that

$$(5.3) \qquad \lim_{x \to -\infty} <x|H_o,A_o;E,a> = 0.$$

In terms of the x-representation Eq. (5.2) becomes

$$(5.4) \qquad -\frac{d^2}{dx^2} < x|H_o,A_o;E,a > \; = \; E < x|H_o,A_o;E,a > \, .$$

For a fixed negative value of E, there is only one solution which satisfies (5.3). We conclude therefore that the spectrum of the extended Hamiltonian H_o and the point spectrum of H cannot be degenerate.

Let us denote by $< x|H_o;E >$ the suitable solutions of (5.4). Then we can write

$$(5.5) \qquad < x|H_o;E > \; = \; A(E) \, \frac{\exp(-i \, \frac{\pi}{4})}{2\sqrt{\pi}} \, (-E)^{-1/4} e^{\sqrt{-E} \, x} \; = \; A(E) < x|H_o,A_o;E,-1 >^{+}$$

where A(E) is an arbitrary function of E which has no zeros or singularities for $E < 0$. With this choice of $|H_o;E >$ we take for the bi-orthogonal vector $|H_o;E >_B$ another solution of (5.2) or (5.4) which satisfies

$$(5.6) \qquad {}_B< H_o,E|H_o,E' > \; = \; \delta(E-E').$$

It is clear that

$$(5.5a) \qquad {}_B< H_o;E|x > \; = \; \left[A(E)\right]^{-1} \frac{\exp(-i \, \frac{\pi}{4})}{2\sqrt{\pi}} \, (-E)^{-1/4} \, e^{-\sqrt{-E}x} \; = \; < H_o,A_o;E,-1|x >^{+} \, .$$

Also

$$(5.5b) \qquad < H_o;E|x > \; = \; A^{*}(E) \, \frac{\exp(i \, \frac{\pi}{4})}{2\sqrt{\pi}} \, (-E)^{-1/4} \, e^{\sqrt{-E}x} \; = \; i \; A^{*}(E) < H_o,A_o;E,+1|x >^{+} .$$

In the general vector space all states $< x|\phi >$ will have the property that they are quadratically integrable from $x = -\infty$ to any finite limit.

We shall now prove V is diagonal in the x-representation. We have

$$\varepsilon V = H - H_0 = UH_0U_0 - H_0UU_0 = (UH_0 - H_0U)U_0 = \varepsilon(KH_0 - H_0K) + \varepsilon^2(KH_0 - H_0K)K_0. \tag{5.7}$$

We shall first calculate $(KH_0 - H_0K)$ in terms of the x-representation. Since K is triangular in the x-representation we can always write

$$< x|K|x' > \; = \eta(x-x') < x|P|x' > , \tag{5.8}$$

where

$$\begin{aligned} < x|P|x' > \; &= \; < x|K|x' > && \text{for } x' \leq x, \\ < x|P|x' > \; &= \; \text{an arbitrary function,} && \text{for } x' > x \end{aligned} \tag{5.9}$$

We choose $< x|P|x' >$ so that it is continuous at $x = x'$. Since H_0 is Hermitian, we have

$$\begin{aligned} < x|KH_0|x' > &= - < x|K|x' > \frac{\partial^2}{\partial x'^2} = - \frac{\partial^2}{\partial x'^2} < x|K|x' > \\ &= - \frac{\partial^2}{\partial x'^2} \eta(x-x') < x|P|x' >. \end{aligned} \tag{5.10}$$

Now

$$\begin{aligned} \frac{\partial}{\partial x'} \eta(x-x') < x|P|x' > &= -\delta(x-x') < x|P|x' > + \eta(x-x') \frac{\partial}{\partial x'} < x|P|x' > \\ &= -\delta(x-x') < x|P|x > + \; \eta(x-x') \frac{\partial}{\partial x'} < x|P|x' >, \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x'^2}\, \eta(x-x') < x|P|x' > &= \delta'(x-x') < x|P|x > - \delta(x-x')\, \frac{\partial}{\partial x'} < x|P|x > \\ &\quad + \eta(x-x')\, \frac{\partial^2}{\partial x'^2} < x|P|x' >. \end{aligned} \tag{5.12}$$

Likewise

$$< x|H_oK|x' > = - \frac{\partial^2}{\partial x^2} < x|K|x' > = - \frac{\partial^2}{\partial x^2}\, \eta(x-x') < x|P|x' > . \tag{5.13}$$

But

$$\frac{\partial}{\partial x}\, \eta(x-x') < x|P|x' > = \delta(x-x') < x|P|x > + \eta(x-x')\, \frac{\partial}{\partial x} < x|P|x' > \tag{5.14}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\, \eta(x-x') < x|P|x' > &= \delta'(x-x') < x|P|x > + \delta(x-x')\, \frac{d}{dx} < x|P|x > \\ &\quad + \delta(x-x')\, \frac{\partial}{\partial x} < x|P|x > + \eta(x-x')\, \frac{\partial^2}{\partial x^2} < x|P|x' >. \end{aligned} \tag{5.15}$$

In (5.15) the expression $\frac{d}{dx} < x|P|x >$ means that after x' has been set equal to x in $< x|P|x' >$, one takes the derivative of $< x|P|x >$ with respect to x. In contrast, the expression $\frac{\partial}{\partial x} < x|P|x >$ means that one has taken the derivative of $< x|P|x' >$ with respect to x and then set x' equal to x. In (5.12) $\frac{\partial}{\partial x'} < x|P|x >$ means that one is to take the derivative of $< x|P|x' >$ with respect to x' and then set x' equal to x. From (5.11) - (5.15), (5.8), and the fact that

$$\frac{\partial}{\partial x} < x|P|x > + \frac{\partial}{\partial x'} < x|P|x > = \frac{d}{dx} < x|P|x > ,$$

one can write

$$\langle x|KH_o-H_oK|x'\rangle = 2\delta(x-x')\,\frac{d}{dx}\langle x|K|x\rangle + \eta(x-x')\left[\frac{\partial^2}{\partial x^2}\langle x|K|x'\rangle - \frac{\partial^2}{\partial x'^2}\langle x|K|x'\rangle\right] \tag{5.16}$$

Hence from (5.7) and the fact that K_o is triangular in the x-representation we have

$$\begin{aligned}\langle x|V|x'\rangle = 2\delta(x-x')\,\frac{d}{dx}\langle x|K|x\rangle + \eta(x-x')\Bigg\{&\frac{\partial^2}{\partial x^2}\langle x|K|x'\rangle - \frac{\partial^2}{\partial x'^2}\langle x|K|x'\rangle\\ &+2\varepsilon\frac{d}{dx}\langle x|K|x\rangle\cdot\langle x|K_o|x'\rangle\\ &+\varepsilon\int_{x'}^{x}\left[\frac{\partial^2}{\partial x^2}\langle x|K|x''\rangle - \frac{\partial^2}{\partial x'^2}\langle x|K|x''\rangle\right]dx''\,\langle x''|K_o|x'\rangle\Bigg\}.\end{aligned} \tag{5.17}$$

Now from the general theory, V is Hermitian if the generalized Gelfand-Levitan equation of [6] can be solved. Furthermore, it will later be shown that $\langle x|K|x'\rangle$ is real. Therefore the right-hand side of (5.17) is the sum of a Hermitian operator which is diagonal in the x-representation, and an operator which, because it is triangular in the x-representation, cannot be Hermitian. Since the right-hand side must be Hermitian, the expression in brackets must vanish and we are left with

$$\langle x|V|x'\rangle = \delta(x-x')\,V(x), \tag{5.18}$$

where

$$(5.19) \qquad V(x) = 2 \frac{d}{dx} < x|K|x > .$$

Hence V is diagonal in the x-representation and can be obtained from K very simply, by means of (5.19).

6. The weight operator and the Gelfand-Levitan equation

We shall now develop expressions for the weight operator in terms of the H_o-representation. In accordance with the general theory, the equation for K is

$$(6.1) \qquad < x|K|x' > = - < x|\Omega|x' > - \varepsilon \int_{-\infty}^{x} < x|K|x'' > dx'' < x''|\Omega|x' >,$$

where

$$(6.2) \qquad \varepsilon\Omega = W - \eta(H_o) = \Big[W_c - \eta(H_o)\Big]\eta(H_o) + W_d \sum_i \delta(E_i - H_o)\,\eta(-H_o).$$

we have already expressed W_c in terms of the H_o-representation and have seen that it can be obtained in terms of the reflection coefficient of the scattering operator (Eqs. (3.14) and (3.16)). We shall now obtain the expression for W_d in terms of the H_o-representation.

From the general procedure discussed in [6], Part II, W_d as given in the H_o-representation has the form

$$(6.3) \qquad {}_B< H_o,A_o;E,a|W_d|H_o,A_o;E',b >_B = \delta(E-E') < a|\omega_d(E)|b > ,$$

where $|H_o,A_o;E,a>_B$ are the bi-orthogonal eigenfunctions (5.5a) of H_o defined for negative eigenvalues E in the neighborhoods of the point eigenvalues E_i of H. Since, in the present case, these eigenfunctions have no degeneracy, we shall write

$$(6.4)\qquad {}_B\langle H_o,E|W_d|H_o,E'\rangle_B = \delta(E-E')\omega_d(E) = \delta(E-E')\,\left[C(E)\right]^{-1},$$

where, in accordance with [6], Part II, the $C(E_i)$ are the normalization constants of the proper eigenstates of H, that is,

$$(6.5)\qquad \langle H,E_i|H,E_j\rangle = C(E_i)\,\delta_{ij}\,.$$

We shall now obtain $\varepsilon\langle x|\Omega|x'\rangle$ in the form

$$(6.6)\qquad \varepsilon\langle x|\Omega|x'\rangle = \langle x|\big(W_c-\eta(H_o)\big)\eta(H_o)|x'\rangle + \langle x|W_d\sum_i\delta(E_i-H_o)\eta(-H_o)|x'\rangle.$$

We note that

$$(6.7)\qquad \langle x|\big(W_c-\eta(H_o)\big)\eta(H_o)|x'\rangle = \sum_{a,b}\int\langle x|H_o,A_o;E,a\rangle\left[\langle a|\omega_c(E)|b\rangle-\delta_{ab}\right]\eta(E)dE\,\langle H_o,A_o;E,b|x'\rangle.$$

On using (3.16) and (2.2) we have

$$(6.8)\qquad \begin{aligned}&\langle x|\big(W_c-\eta(H_o)\big)\eta(H_o)|x'\rangle\\ &= \int_0^\infty dE\,\langle x|H_o,A_o;E,-1\rangle\langle -1|S(E)|+1\rangle\langle H_o,A_o;E,+1|x'\rangle\\ &\quad+\int_0^\infty dF\,\langle x|H_o,A_o;E,+1\rangle\langle -1|S(E)|+1\rangle^*\langle H_o,A_o;E,-1|x'\rangle\\ &= 2\,\mathrm{Re}\int_0^\infty dE\,\langle x|H_o,A_o;E,-1\rangle\langle -1|S(E)|+1\rangle\langle H_o,A_o;E,+1|x'\rangle,\end{aligned}$$

where Re means 'real part'. Eq. (6.8) leads to

$$< x|(W_c - \eta(H_o))\eta(H_o)|x' > = \mathrm{Re}\,\frac{1}{2\pi}\int_0^\infty dE\; E^{-\frac{1}{2}}\exp\left[-i\sqrt{E}(x+x')\right]< -1|S(E)|+1 >. \tag{6.9}$$

We can simplify (6.9) by introducing the momentum variable defined by

$$E = k^2 . \tag{6.10}$$

Then

$$< x|(W_c - \eta(H_o))\eta(H_o)|x' > = \mathrm{Re}\,\frac{1}{\pi}\int_0^\infty dk\; e^{-ik(x+x')} < -1|S(k^2)|+1 >. \tag{6.11}$$

We also have

$$\begin{aligned} < x|W_d\sum_i \delta(E_i - H_o)\eta(-H_o)|x' > &= \sum_i \int dE < x|H_o;E > \left[C(E)\right]^{-1}\delta(E_i - E) < H_o;E|x' > \\ &= \sum_i < x|H_o;E_i > \left[C(E_i)\right]^{-1} < H_o;E_i|x' >, \end{aligned} \tag{6.12}$$

where the integration is over the intervals ΔE_i. On using (5.5) we have

$$\begin{aligned} < x|W_d\sum_i \delta(E_i - H_o)\eta(-H_o)|x' > &= \frac{1}{4\pi}\sum_i (-E_i)^{-\frac{1}{2}}\exp\left[\sqrt{-E_i}(x+x')\right]\left[C(E_i)\right]^{-1}|A(E_i)|^2 \\ &= i\sum_i \left[C(E_i)\right]^{-1}|A(E_i)|^2 < x|H_o,A_o;E_i,-1 >^+ \\ &\qquad < H_o,A_o;E_i,+1|x' >^+ . \end{aligned} \tag{6.13}$$

Hence finally we have

$$
\begin{aligned}
\varepsilon < x|\Omega|x' > &= 2 \operatorname{Re} \int_0^\infty dE < x|H_o, A_o; E, -1 > < -1|S(E)|+1 > < H_o, A_o; E, +1|x' > \\
&\quad + i \sum_i \left[C(E_i)\right]^{-1} |A(E_i)|^2 < x|H_o, A_o; E_i, -1 >^+ < H_o, A_o; E_i, +1|x' >^+ \\
&= \operatorname{Re} \frac{1}{\pi} \int_0^\infty dk\, e^{ik(x+x')} < -1|S(k^2)|+1 > + \frac{1}{4\pi} \sum_i (-E_i)^{-\frac{1}{2}} \left[C(E_i)\right]^{-1} |A(E_i)|^2 \\
&\qquad \exp\left[\sqrt{-E_i}\,(x+x')\right] .
\end{aligned}
\tag{6.14}
$$

Clearly $< x|\Omega|x' >$ is real. Then from the uniqueness theorem it can be shown that $< x|K|x' >$ is also real, as stated in Section 5.

7. The conditions on the bound states and the reflection coefficient.

In accordance with the general theory of [6], we can generally find the operators $U = I + \varepsilon K$, $U_o = I + \varepsilon K_o$, and therefore the complete set of eigenfunctions $|H, A; E, a >$ $= U|H_o, A_o; E, a >$ and $|H; E_i > = U|H_o; E_i >$ corresponding to the continuous and the discrete spectrum of $H = UH_oU_o$ respectively whose normalization is given by the matrix elements of $W = \eta(H_o) + \varepsilon\Omega$, where Ω is given by (6.14); these eigenfunctions are

$$
\begin{aligned}
< H, A; E, a|H, A; E', a' > &= < H_o, A_o; E, a|W^{-1}\eta(H_o)|H_o, A_o; E', a' > \\
&= < a|\omega^{-1}(E)|a' > \delta(E-E')\eta(E),
\end{aligned}
\tag{7.1}
$$

$$
< H, E_i|H, E_j > = \delta_{ij}\, C_i ,
\tag{7.2}
$$

$$
< H, A; E, a|H, E_i > = 0.
\tag{7.3}
$$

However, in order to be interesting, our solution must be such that we are able to construct a scattering of S in which the reflection coefficient $< -1|S(E)|+1 >$ is the one used to form the weight operator W. Toward this end we shall obtain some necessary conditions on the reflection coefficient, point eigenvalues, and normalization constants.

The necessary conditions follow from the requirements on the rapidity with which V(x) must decay as x approaches $+\infty$ or $-\infty$.

Let us first discuss consequences of the decay of V(x) as $x \to -\infty$. If V(x) is to decay sufficiently rapidly to ensure the existence of a scattering operator, one can show that the outgoing wave $< x|H,A;E,a >_-$ must be unique. As a consequence of this uniqueness it follows that we must have

$$(7.4) \qquad < x|H,A;E,a > \;\to\; < x|H_0,A_0;E,a > \quad \text{as } x \to -\infty.$$

In [6], Part I, Section 17 we have already discussed two requirements on $< x|\Omega|x' >$ which ensure the validity of (7.4). However, in the present application of the general procedure, it turns out that because $< x|\Omega|x' >$ is a function of $x + x'$ (see (6.14)) another approach is useful. A sufficient condition for the validity of (7.4) is that a positive number α exists such that

$$(7.5) \qquad < x|K|x' > \;\equiv\; 0, \qquad x < -\alpha.$$

It is clear that a consequence of (7.5) is that

$$(7.6) \qquad V(x) = 2\frac{d}{dx} < x|K|x > \;\equiv\; 0, \qquad \text{for } x < -\alpha,$$

i.e., that V(x) is cut off from below.

We shall now show that a necessary and sufficient condition for (7.5) to hold is

$$(7.7) \qquad \langle x|\Omega|0\rangle \equiv 0, \qquad x < -2\alpha,$$

and that a necessary condition for (7.7) and hence (7.5) to be valid is that the point eigenvalues E_i be the poles of $\langle -1|S(E)|+1\rangle^+$, i.e., the analytic continuation of the reflection coefficient. A sufficient condition for (7.7) and (7.5) to hold is that $\langle +1|S(E)|-1\rangle^+$ be of the form $e^{-i2\sqrt{E}\,\alpha} f(E)$ where $f(E)$ behaves like $O(E^{-1/2})$ for $|E| \to \infty$ in the cut complex plane.

Let us first prove that (7.7) is a necessary condition for (7.5) to hold. From (6.14) the following identity is obvious:

$$(7.8) \qquad \langle x|\Omega|x'\rangle = \langle x+x'|\Omega|0\rangle .$$

The Gelfand-Levitan equation (6.1) can then be written

$$(7.9) \qquad \langle x|K|x'\rangle = -\langle x+x'|\Omega|0\rangle - \varepsilon\int_{-\infty}^{x}\langle x|K|x''\rangle\, dx''\,\langle x''+x'|\Omega|0\rangle \quad (x > x').$$

Now let $x < -\alpha$. From (7.5) we have

$$(7.10) \qquad \langle x+x'|\Omega|0\rangle \equiv 0 \qquad \text{for } x < -\alpha, \quad x' < x < -\alpha,$$

from which (7.7) follows as a necessary condition.

Let us now show the sufficiency of condition (7.7). If we use Eq. (7.7), then Eq. (7.9) becomes

$$(7.11)\quad < x|K|x' > = - < x+x' |\Omega|0 > - \varepsilon\, \eta\,(x+2\alpha+x') \int_{-(2\alpha+x')}^{x} < x|K|x'' > dx'' < x''+ x' |\Omega|0 >, \qquad (x < x').$$

When $x < -\alpha$, we have $x + x' < -2\alpha$ or, equivalently, $x < -(2\alpha + x')$; hence the right-hand side of (7.11) vanishes and we are left with (7.5).

To show the connection between the singularities of $< -1|S(E)| +1 >$ and the bound states we note that we can write the expression for $\varepsilon < x|\Omega|x' >$ as an integral in the complex E-plane. On using (4.11), which gives the analytic continuation of the matrix elements of the scattering operator, we see that

$$< -1|S(E)|+1 >^{+} = < -1|S(E)|+1 >^{*}$$

where E is at the bottom of the cut.

Hence, using this fact and the fact that

$$(7.12)\quad < x|H_o,A_o;\, E,-1 >^{+} < H_o,A_o;\, E,+1|x' >^{+} = \frac{E^{-1/4}}{2\sqrt{\pi}} < x+x' |H_o,A_o;\, E,-1 >^{+} ,$$

we have

$$\varepsilon < x|\Omega|x' >$$

$$(7.13)\quad = \frac{1}{2\sqrt{\pi}} \int_{\gamma} < -1|S(E)|+1 >^{+} (E)^{-1/4} < x+x' |H_o,A_o;\, E,-1 >^{+} dE + \frac{1}{2\sqrt{\pi}}\, e^{i\pi/4} \sum_{i} \left[C(E_i)\right]^{-1} |A(E_i)|^{2} (-E_i)^{-1/4} < x+x' |H_o,A_o;\, E_i,-1 >^{+},$$

where the contour γ goes in a clockwise direction around the cut in the E-plane.

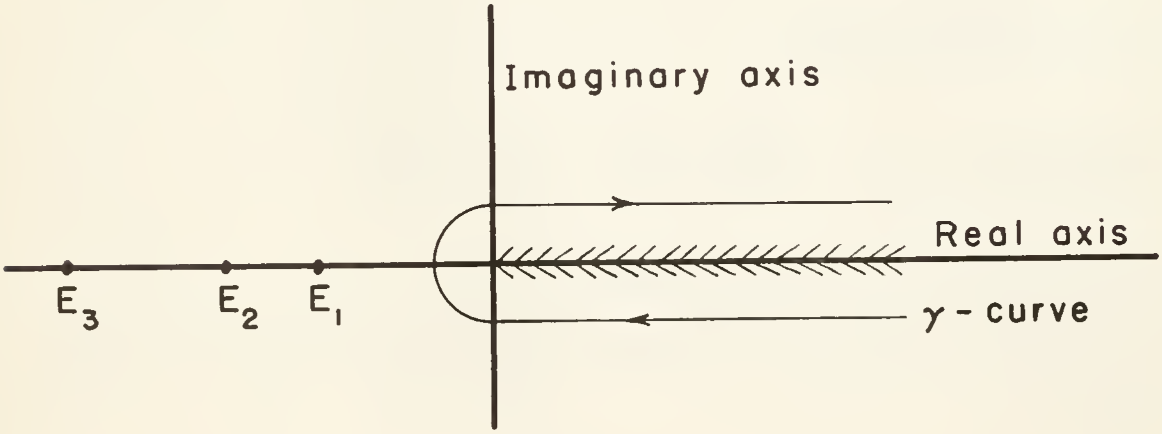

Fig. 1

Complex E-plane

Let us introduce the momentum variable $k = \sqrt{E}$. The E-plane is then transformed to the upper k-plane. Let us write $b(k) = < -1|S(k^2)| + 1 >$. From (4.11) it is clear that $b(-k) = b^*(k)$ for real k. We have

$$\varepsilon < x|\Omega|0 > = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\; b(k)e^{-ikx}\, dk$$

$$+ \frac{1}{2\sqrt{\pi}}\, e^{i\pi/4} \sum_i \left[C(E_i)\right]^{-1} |A(E_i)|^2 (-E_i)^{-1/4} < x|H_o, A_o; E_i, -1 >^+ .$$

If we now assume that $< -1|S(E)| + 1 > \to e^{-i2\sqrt{E}\,a} f(E)$ as $|E| \to \infty$, where $f(E) \to O(E^{-1/2})$, we have

$$(7.15) \qquad b(k) \to e^{-i2ka}\, g(k) \qquad |k| \to \infty,$$

where $g(k) = f(E) \to O(k^{-1})$ as $|k| \to \infty$.

The integral of Eq. (7.14) becomes

$$(7.16) \qquad \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(k)e^{-ik(x + 2a)}\, dk \quad .$$

When $x < -2a$, one can close the contour by the infinite semi-circle in the upper half of the k-plane, since the integral over this semi-circle will contribute nothing.

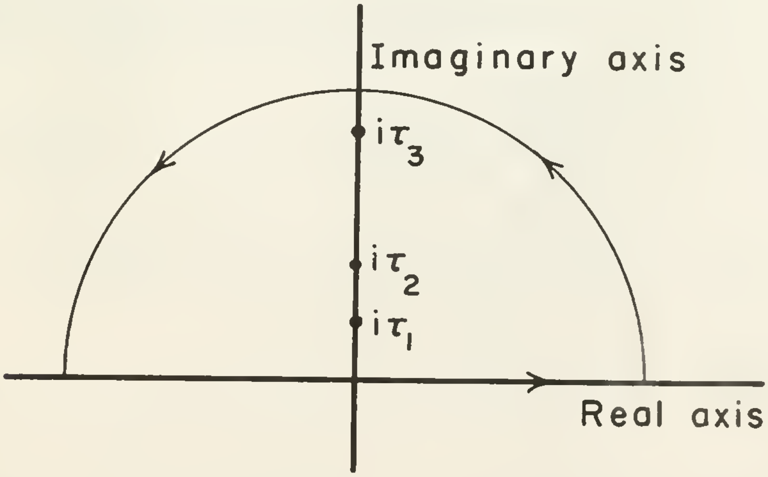

Fig. 2

Complex k-plane

The integral (7.16) is equal to the sum of the residues of the integrand, namely,

$$(7.17) \qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(k)e^{-ik(x+2\alpha)}dk = i\sum_j r_j\, e^{-2ik_j\alpha}\, e^{-ik_jx} ,$$

where r_j are the resides of $g(k)$ at the poles k_j. Since the integral must be real for all x, it is clear that the poles k_j must lie on the positive imaginary axis. We shall write $k_j = i\tau_j$ where τ_j are positive real numbers see (Fig. 2). Likewise, ir_j must be real.

Finally, then, we can write for $x < -2\alpha$

$$(7.18) \qquad \begin{aligned} \varepsilon\langle x|\Omega|0\rangle &= 2\sqrt{\pi}\, e^{i\pi/4}\, i\sum_j r_j e^{2\tau_j\alpha}(-E_j)^{1/4}\langle x|H_0,A_0;\, E_j,-1\rangle^+ \\ &+ \frac{1}{2\sqrt{\pi}}\, e^{i\pi/4}\sum_i \left[C(E_i)\right]^{-1}|A(E_i)|^2(-E_i)^{-1/4}\langle x|H_0,A_0;\, E_i,-1\rangle^+ \end{aligned}$$

where in the first sum $E_j = k_j^2 = -\tau_j^2$ (see Fig. 1).

Since $\langle x|\Omega|0\rangle$ is to vanish for $x < -2\alpha$ it is clear that it is sufficient to require that the poles k_j of $g(k)$ be equal to $i\sqrt{-E_j}$, where E_j are the point eigenvalues of H, and that

$$(7.19) \qquad \left[C(E_i)\right]^{-1}|A(E_i)|^2 = -4\pi i\,\tau_i\, r_i \exp\left[2\tau_i\alpha\right] .$$

From (7.19) it is seen that r_i must be a positive imaginary number in order that $C(E_i)$ be positive.

On using (7.19) in (7.14) we obtain

$$(7.14a) \qquad \varepsilon\langle x|\Omega|0\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, b(k)\, e^{-ikx} - i\sum_j r_j\, e^{\tau_j(x+2\alpha)} .$$

One then obtains the results in terms of the notation of the introduction by writing

$$\langle x|H,A;\, E,a\rangle = \frac{1}{2\sqrt{\pi}}\, E^{-1/4}\, \Psi_k(x)$$

where $k = a\sqrt{E}$.

We have given sufficient conditions for the reproduction of the reflection coefficient $\langle -1|S(E)|+1\rangle$, in particular, the condition that $V(x) = 0$ for

$x < -\alpha$. However, we are not as yet able to obtain conditions to guarantee the vanishing of $V(x)$ for x sufficiently large. This vanishing is a necessary condition for the full scattering operator to exist. It is, in fact, possible to give examples where the prescribed reflection coefficient exists, but where the full scattering operator does not exist.

Incidentally, the use of a triangular operator $< x|K|x' >$ which vanishes when $x < -\alpha$ has the consequence that $< x|H,A;\, E,a > = < x|U|H_o,A_o;\, E,a >$ is an analytic function of E in the complex E-plane. Furthermore, the bound states $< x|H,E_i >$ can be obtained from the analytic continuation of $< x|H,A;\, E,-1 >$. In fact

$$(7.20) \qquad < x|H,A;\, E_i,-1 >^+ = < x|U|H_o,A_o;\, E_i,-1 >^+ = \left[A(E_i)\right]^{-1} < x|H,E_i > \ .$$

Usually one chooses the arbitrary function $A(E_i)$ to be unity, to simplify the relation (7.20). However, to obtain the results in terms of the notation of the introduction, we write

$$A(E_i) = 2\sqrt{\pi}\ \exp(i\pi/4)\ (-E_i)^{1/4} \ .$$

8. Examples

We shall now discuss three examples. In what follows we shall take $\varepsilon = 1$ for simplicity.

A) The repulsive δ-function potential

Let us consider the case where

$$(8.1) \qquad < -1|S(E)| +1 > = \frac{-\, iA \exp\left[-2i\sqrt{E}\,\alpha\right]}{2\,E^{1/2} + i\,A} \ , \qquad (A > 0)$$

or equivalently

$$(8.2) \qquad b(k) = \frac{-\, iA/2 \exp\left[-\,2ik\alpha\right]}{k + \frac{iA}{2}} \ .$$

We see that $b(-k) = b^*(k)$ (for real k) as required. Furthermore, $b(k) = O(k^{-1})$ for $|k| \to \infty$. Since $b(k)$ has no poles on the positive imaginary axis, there will be no bound states.

From (1.8) we have

$$
\begin{aligned}
\langle x|\Omega|x'\rangle &= \langle x + x'|\Omega|0\rangle \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\; b(k)\; e^{-ik(x + x')} \\
&= \frac{-iA}{4\pi}\int_{-\infty}^{+\infty} \frac{dk}{k + \frac{iA}{2}}\; e^{-ik(x + x' + 2\alpha)} \\
&= \frac{-A}{2}\,\eta(x + x' + 2\alpha)\, e^{-\frac{A}{2}(x + x' + 2\alpha)} \quad .
\end{aligned} \tag{8.3}
$$

The Gelfand-Levitan equation (1.7) is

$$
\begin{aligned}
\langle x|K|x'\rangle = \frac{A}{2}\eta(x+x'+2\alpha)e^{-\frac{A}{2}(x+x'+2\alpha)} + \frac{A}{2}\quad(x+x'+2\alpha)\int_{-(2\alpha+x')}^{x} \langle x|K|x''\rangle \\
\cdot\ \eta(x'' + x' + 2\alpha)\, e^{-\frac{A}{2}(x'' + x' + 2\alpha)}\, dx'' \quad .
\end{aligned} \tag{8.4}
$$

It is easily verified that the solution of (8.4) is

$$\langle x|K|x'\rangle = \frac{A}{2}\,\eta(x + x' + 2\alpha) \qquad x' < x \, . \tag{8.5}$$

Hence on using the potential V(x) is

$$V(x) = 2\,\frac{d}{dx}\,\frac{A}{2}\,\eta(2x + 2\alpha) = A\,\delta(x + \alpha) \quad . \tag{8.6}$$

It is clear that $V(x) \equiv 0$ and $\langle x|K|x'\rangle \equiv 0$ when $x < -\alpha$ as required.

The eigenfunctions of $H = -d^2/dx^2 + A\,\delta(x + \alpha)$ are given by

$$
\begin{aligned}
\langle x|H,A;E,a\rangle &= \langle x|H_o,A_o;E,a\rangle + \frac{A}{2}\int_{-\infty}^{+\infty}\eta(x+x'+2\alpha)\,\langle x'|H_o,A_o;E,a\rangle\, dx' \\
&= \langle x|H_o,A_o;E,a\rangle + \frac{A}{2}\eta(x+\alpha)\int_{-(x+2\alpha)}^{x}\langle x'|H_o,A_o;E,a\rangle\, dx' \\
&= \langle x|H_o,A_o;E,a\rangle - \frac{iAa}{2}\eta(x+\alpha)E^{-1/2} \\
&\qquad \cdot \Big[\langle x|H_o,A_o;E,a\rangle - \langle x+2\alpha|H_o,A_o;E,-a\rangle\Big]
\end{aligned} \tag{8.7}
$$

We shall now calculate the scattering operator for the problem. Toward this end we shall use

$$S = M_+ M_-^{-1} \tag{8.8}$$

(Cf. [6], Part I, Eq. (5.11)) or equivalently

$$\langle a|S(E)|b\rangle = \sum_c \langle a|\mu_+(E)|c\rangle\langle c|\mu_-^{-1}(E)|b\rangle \quad . \tag{8.9}$$

Hence we shall have to evaluate $\langle a|M_+|b\rangle$ and $\langle a|M_-^{-1}(E)|b\rangle$.

From [6], Part I, Eq. (5.1) we have

$$M_{\pm} = U - \int \gamma_{\pm}(E - H_0)V\,U\,\delta(E - H_0)\,dE \quad . \tag{8.10}$$

If there were bound states, the above equation would have to be replaced by

$$M_{\pm}\,\eta(H_0) = U\,\eta(H_0) - \int \gamma_{\pm}(E - H_0)V\,U\,\eta(H_0)\delta(E - H_0)dE \quad . \tag{8.11}$$

Equation (8.10) is equivalent to

$$\begin{aligned} &\sum_{a'} \langle x|H_0,A_0;\ E,a'\rangle\langle a'|\mu_{\pm}(E)|a\rangle \\ &= \langle x|H,A;E,a\rangle \mp \frac{i}{2}(E)^{-1/2}\int_{-\infty}^{+\infty} \exp\left[\mp i\sqrt{E}\ |x-x'|\right]V(x')\langle x'|H,A;E,a\rangle\,dx' \quad . \end{aligned} \tag{8.12}$$

In obtaining (8.12) from (8.10) we used Eqs. (3.10) and (2.7a).

On using (8.6) and (8.7) we have for $x < -\alpha$

$$\begin{aligned} &\sum_{a'} \langle x|H_0,A_0;\ E,a'\rangle\langle a'|\mu_{\pm}(E)|a\rangle \\ &= \langle x|H_0,A_0;\ E,a\rangle \mp \frac{Ai}{2}(E)^{-1/2}\exp\left[\pm i\sqrt{E}\ (x+\alpha)\right]\langle -\alpha|H_0,A_0;\ E,a\rangle \end{aligned} \tag{8.13}$$

or

$$\begin{aligned} &\sum_{a'} \langle x|H_0,A_0;\ E,a'\rangle\langle a'|\mu_{\pm}(E)|a\rangle \\ &= \langle x|H_0,A_0;\ E,a\rangle \mp \frac{Ai}{2}E^{-1/2}\exp\left[-i\alpha(a\mp 1)\sqrt{E}\right]\langle x|H_0,A_0;\ E,\pm 1\rangle \quad . \end{aligned} \tag{8.14}$$

Since $< a|\mu_{\pm}(E)|a' >$ must be independent of x, we have

$$(8.15)\qquad < a|\mu_{-}(E)|a' > = \begin{pmatrix} 1 & 0 \\ \dfrac{Ai\ E^{-1/2}}{2}\, e^{-2i\alpha\sqrt{E}} & 1 + \dfrac{Ai\ E^{-1/2}}{2} \end{pmatrix},$$

$$(8.16)\qquad < a|\mu_{+}(E)|a' > = \begin{pmatrix} 1 - \dfrac{Ai\ E^{-1/2}}{2} & - \dfrac{Ai\ E^{-1/2}}{2}\, e^{2i\alpha\sqrt{E}} \\ 0 & 1 \end{pmatrix}.$$

Since the matrix $< a|\mu_{-}^{-1}(E)|a' >$ is just the reciprocal of $< a|\mu_{-}(E)|a' >$ we see that

$$(8.17)\qquad < a|\mu_{-}^{-1}(E)|a' > = \begin{pmatrix} 1 & 0 \\ \dfrac{- Ai\ e^{-2i\alpha\sqrt{E}}}{2\sqrt{E} + i\,A} & \dfrac{2\sqrt{E}}{2\sqrt{E} + iA} \end{pmatrix}.$$

Finally on using (8.9)

$$(8.18)\qquad < a|S(E)|a' > = \begin{pmatrix} \dfrac{2\sqrt{E}}{2\sqrt{E} + i\,A} & \dfrac{- i\,A\,e^{2i\alpha\sqrt{E}}}{2\sqrt{E} + i\,A} \\ \dfrac{- Ai\ e^{-2i\alpha\sqrt{E}}}{2\sqrt{E} + i\,A} & \dfrac{2\sqrt{E}}{2\sqrt{E} + i\,A} \end{pmatrix}.$$

The reflection coefficient $< -1|S(E)| + 1 > = \dfrac{- Ai\ e^{-2i\alpha\sqrt{E}}}{2\sqrt{E} + i\,A}$ is just the one we started with.

B) The attractive δ-function potential

Let us now consider the reflection coefficient

$$(8.19) \qquad < -1|S(E)| +1 > \; = \; \frac{i\,B\,e^{-2i\alpha\sqrt{E}}}{2\sqrt{E} - i\,B} \; , \qquad B > 0$$

or

$$(8.20) \qquad b(k) \; = \; \frac{iB/2\; e^{-2i\alpha k}}{k - \frac{i\,B}{2}} \; .$$

In this case there is a bound state, since b(k) has a singularity on the positive imaginary axis. In fact we have

$$(8.21) \qquad \begin{cases} \gamma_1 = \frac{B}{2} \\ r_1 = \frac{iB}{2} \end{cases}$$

and the bound state E_1 is given by

$$(8.22) \qquad E_1 = -\gamma_1^2 = -\frac{B^2}{4} \; .$$

We have

$$(8.23) \qquad \begin{aligned} < x|\Omega|x' > \; &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k)\, e^{-ik(x+x')}dk - i\, r_1\, e^{\gamma_1(x+x'+2\alpha)} \\ &= \frac{iB}{4\pi}\int_{-\infty}^{+\infty} \frac{e^{-ik(x+x'+2\alpha)}}{k - \frac{iB}{2}}\, dk + \frac{B}{2}\, e^{\frac{B}{2}(x+x'+2\alpha)} \\ &= -\frac{B}{2}\,\eta(-x-x'-2\alpha)\, e^{\frac{B}{2}(x+x'+2\alpha)} + \frac{B}{2}\, e^{\frac{B}{2}(x+x'+2\alpha)} \\ &= \frac{B}{2}\,\eta(x+x'+2\alpha)\; e^{\frac{B}{2}(x+x'+2\alpha)} \end{aligned}$$

The solution of the Gelfand-Levitan equation is

$$(8.24) \qquad < x|K|x' > \; = \; -\frac{B}{2}\,\eta(x+x'+2\alpha)$$

and the potential V(x) is

$$V(x) = 2 \frac{d}{dx} < x|K|x > = - B\, \delta(x + \alpha) \quad . \tag{8.25}$$

The eigenfunctions of the continuous spectrum are

$$\begin{aligned} < x|H,A;\ E,a > &= < x|H_o,A_o;\ E,a > + \frac{iBa}{2} \eta(x+\alpha)(E)^{-1/2} \\ &\cdot \Big[< x|H_o,A_o;\ E,a > - < x+2\alpha|H_o,A_o;\ E,-a > \Big] \end{aligned} \tag{8.26}$$

while the proper eigenfunction corresponding to the eigenvalue $E_1 = -\frac{B^2}{4}$ is on using (7.20) and (5.5)

$$\begin{aligned} < x|H,E_1 > &= A(E_1) < x|H,A;\ E_1,-1 >^+ \\ &= \frac{A(E_1)}{2\sqrt{\pi}} \exp\left[-\frac{i\pi}{4} - \frac{B}{2}\alpha \right] \exp\left[-\left|\frac{B}{2}(x+\alpha)\right| \right] \end{aligned} \tag{8.27}$$

where, in accordance with (7.19) and (8.21)

$$\left[C(E_1)\right]^{-1} |A(E_1)|^2 = \frac{4\pi B^2}{4} \exp[B\alpha] \quad . \tag{8.28}$$

We can choose wither $A(E_1)$, in which case the normalization $C(E_1)$ is determined or $C(E_1)$ in which case $A(E_1)$ is determined to within a phase factor.

It can also be shown that

$$< a|\mu_-(E)|a' > = \begin{pmatrix} 1 & 0 \\ \frac{-Bi\,E^{-1/2}\, e^{-2i\alpha\sqrt{E}}}{2} & 1 - \frac{Bi\,E^{-1/2}}{2} \end{pmatrix} , \tag{8.29}$$

$$< a|\mu_+(E)|a' > = \begin{pmatrix} 1 + \frac{Bi\,E^{-1/2}}{2} & \frac{B}{2} i\, E^{-1/2}\, e^{2i\alpha\sqrt{E}} \\ 0 & 1 \end{pmatrix} , \tag{8.30}$$

$$(8.31) \qquad \langle a|\mu_-^{-1}(E)|a'\rangle = \begin{pmatrix} 1 & 0 \\ \dfrac{Bi\; e^{-2i\alpha\sqrt{E}}}{2\sqrt{E} - i\,B} & \dfrac{2\sqrt{E}}{2\sqrt{E} - i\,B} \end{pmatrix},$$

$$(8.32) \qquad \langle a|S(E)|a'\rangle = \begin{pmatrix} \dfrac{2\sqrt{E}}{2\sqrt{E} - i\,B} & \dfrac{iB\; e^{2i\alpha\sqrt{E}}}{2\sqrt{E} - i\,B} \\ \dfrac{Bi\; e^{-2i\alpha\sqrt{E}}}{2\sqrt{E} - i\,B} & \dfrac{2\sqrt{E}}{2\sqrt{E} - i\,B} \end{pmatrix}.$$

C) A more complicated example

We shall now summarize the results of a more complicated example discussed in more detail in [7].

Let us find the potential whose reflection coefficient is given by

$$(8.33) \qquad \langle -1|S(E)|+1\rangle = -\exp\left[-2i\alpha\sqrt{E}\right]\frac{1}{E+1}.$$

or

$$(8.34) \qquad b(k) = -e^{-i2\alpha k}\,\frac{1}{(k+i)(k-i)}.$$

There is only one pole in the upper k-plane, namely $k_1 = i$, and hence we can calculate the quantities associated with the bound states:

$$(8.35) \qquad \begin{cases} \tau_1 = 1, \\ r_1 = \dfrac{i}{2}, \\ E_1 = -1, \\ C^{-1}(E_1)|A(E_1)|^2 = 2\pi\, e^{2\alpha}. \end{cases}$$

Also

$$(8.36) \qquad \langle x|\Omega|x'\rangle = \eta(x+x'+2\alpha)\,\sinh(x+x'+2\alpha).$$

Then it can be shown that the solution of the Gelfand-Levitan equation is

$$(8.37) \qquad < x|K|x' > = -\frac{1}{\sqrt{2}}\,\eta(x+x'+2\alpha)\,\frac{\sinh\sqrt{2}\,(x+\alpha) + \sinh\sqrt{2}\,(x'+\alpha)}{\cosh\sqrt{2}\,(x+\alpha)} \; .$$

The potential V(x) is

$$(8.38) \qquad V(x) = 2\,\frac{d}{dx} < x|K|x > = -4\,\eta(x+\alpha)\,\mathrm{sech}^2\sqrt{2}\,(x+\alpha) \; .$$

The eigenfunctions of the continuous spectrum are given by

$$< x|H,A;\; E,a > = \frac{(E)^{-1/4}}{2\sqrt{\pi}}\,\exp\left[ia\sqrt{E}\,x\right] - \frac{\eta(x+\alpha)}{\sqrt{2\pi}}\,\frac{(E)^{-3/4}}{(E+2)}\,\exp\left[-is\sqrt{E}\,\alpha\right]$$

$$(8.39) \qquad \cdot \Big\{\tanh\sqrt{2}(x+\alpha)\left[a(E+2)\sin a\sqrt{E}(X+\alpha) - ia\,E\cos a\sqrt{E}(x+\alpha)\right]$$

$$+ \sqrt{2}\,i\sqrt{E}\,\sin a\sqrt{E}(x+\alpha)\Big\} \; .$$

The single eigenstate corresponding to a point eigenvalue is given by

$$(8.40) \qquad < x|H,E_1 > = A(E_1)e^{-i\pi/4}\left\{\frac{e^{x}}{2\sqrt{\pi}} + \frac{\eta(x+\alpha)}{\sqrt{2\pi}}\,e^{-\alpha}\left[\tanh\sqrt{2}(x+\alpha)e^{-(x+\alpha)} - \sqrt{2}\sinh(x+\alpha)\right]\right\} .$$

It is readily verified that $\lim_{x \to +\infty} < x|H,E_1 > = 0$.

We also derive using the methods of Part A

$$(8.41) \qquad < a|\mu_-(E)|a' > = \begin{pmatrix} 1 & 0 \\ \dfrac{\exp\left[-2i\sqrt{E}\,\alpha\right]}{\sqrt{E}\,(\sqrt{E} + i\sqrt{2})} & \dfrac{E+1}{\sqrt{E}\,(\sqrt{E} + i\sqrt{2})} \end{pmatrix} ,$$

$$(8.42) \qquad < a|\mu_-^{-1}(E)|a' > = \begin{pmatrix} 1 & 0 \\ \dfrac{-\exp\left[-2i\sqrt{E}\,\alpha\right]}{(E+1)} & \dfrac{\sqrt{E}\,(\sqrt{E} + i\sqrt{2})}{(E+1)} \end{pmatrix} ,$$

$$(8.43) \qquad < a|\mu_{+}(E)|a' > = \begin{pmatrix} \dfrac{(E+1)(\sqrt{E} + i\sqrt{2})}{\sqrt{E}\,(E+2)} & \dfrac{\exp\left[2i\alpha\sqrt{E}\right](\sqrt{E} + i\sqrt{2})}{\sqrt{E}\,(E+2)} \\ 0 & 1 \end{pmatrix},$$

$$(8.44) \qquad < a|S(E)|a' > = \begin{pmatrix} \dfrac{\sqrt{E}\,(\sqrt{E} + i\sqrt{2})}{(E+1)} & \dfrac{\exp\left[2i\alpha\sqrt{E}\right](\sqrt{E} + i\sqrt{2})^{2}}{(E+2)\,(E+1)} \\ \dfrac{\exp\left[-2i\alpha\sqrt{E})\right]}{(E+1)} & \dfrac{\sqrt{E}\,(\sqrt{E} + i\sqrt{2})}{(E+1)} \end{pmatrix}.$$

In [7] it is shown how one may solve the Gelfand-Levitan equation when one is given any reflection coefficient which is a rational function of $\sqrt{E}$. In such a case, the Gelfand-Levitan equation reduces to a system of algebraic equations for which a procedure for solving them can be found. The method is analogous to that used by V. Bargmann for the radial equation.

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The determination of the
scattering potential from
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Part III.

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